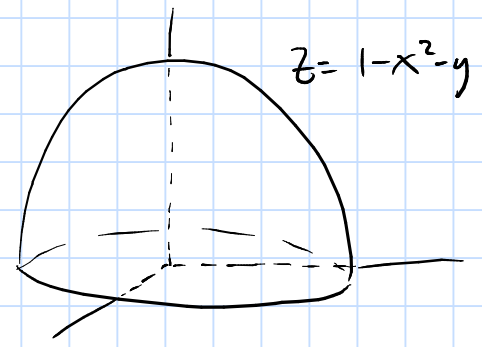
Ex: Compute $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ for $\vec{F} = \langle x^2 \sin(z), y^2, xy \rangle$ and S the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane oriented upward.

Picture



Sol 1 (Direct Computation, xy-parameterization):

$\vec{S}(x,y) = \langle x, y, 1 - x^2 - y^2 \rangle$ on $\{(x,y) : x^2 + y^2 \le 1\}$

$\vec{S}_x = \langle 1, 0, -2x \rangle$

$\vec{S}_y = \langle 0, 1, -2y \rangle$

$\therefore \vec{S}_x \times \vec{S}_y = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{bmatrix} = \langle 0 - (-2x), -(-2y - 0), 1 - 0 \rangle = \langle 2x, 2y, 1 \rangle$ ← oriented upward ☺

$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 \sin(z) & y^2 & xy \end{bmatrix} = \langle x - 0, -(y - x^2 \cos(z)), 0 - 0 \rangle$

$= \langle x, x^2 \cos(z) - y, 0 \rangle$

$\text{curl}(\vec{F})(\vec{S}(x,y)) = \langle x, x^2 \cos(1 - x^2 - y^2) - y, 0 \rangle$

$\therefore \text{curl}(\vec{F})(\vec{S}(x,y)) \cdot (\vec{S}_x \times \vec{S}_y) = 2x^2 + (2x^2 y \cos(1 - x^2 - y^2) - 2y^2) + 0$

So $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_D (2x^2 - 2y^2 + 2x^2 y \cos(1 - x^2 - y^2)) \, dA$ ← $D_{x,y} = \{(x,y) : x^2 + y^2 \le 1\}$

use polar transformation!

$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$ $dA_{xy} = r \, dA_{r\theta}$

$D_{r,\theta} = \{(r,\theta) : 0 \le r \le 1, \ 0 \le \theta \le 2\pi\}$

$= 2 \int_{r=0}^{1} \int_{\theta=0}^{2\pi} (r^2 \cos^2(\theta) - r^2 \sin^2(\theta) + r^3 \cos^2(\theta) \sin(\theta) \cos(1 - r^2)) \, r \, d\theta \, dr$

$= 2 \int_{r=0}^{1} r^3 \int_{\theta=0}^{2\pi} ((\cos^2(\theta) - \sin^2(\theta)) + r\cos(1 - r^2) \cos^2(\theta) \sin(\theta)) \, d\theta \, dr$

$x^2 + y^2 = r^2$

Inner Integral: Note $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$.

$\int_{\theta=0}^{2\pi} ((\cos^2(\theta) - \sin^2(\theta)) + r\cos(1 - r^2) \cos^2(\theta) \sin(\theta)) \, d\theta$

$= \int_{\theta=0}^{2\pi} \cos(2\theta) \, d\theta + r\cos(1 - r^2) \int_{\theta=0}^{2\pi} \cos^2(\theta) \sin(\theta) \, d\theta$ ← $u = \cos(\theta)$, $du = -\sin(\theta) \, d\theta$, $u(2\pi) = 1$, $u(0) = 1$

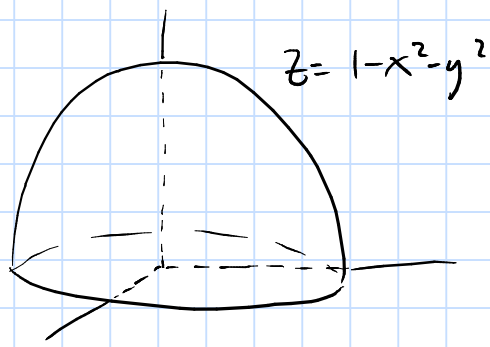
$= \left[\frac{1}{2} \sin(2\theta) \right]_{\theta=0}^{2\pi} + r\cos(1 - r^2) \int_{u=1}^{1} -u^2 \, du$

$= \frac{1}{2}(0 - 0) + r\cos(1 - r^2) \cdot 0 = 0$

$\therefore \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 2 \int_{r=0}^{1} 0 \, dr = 0$ ∎

Ex: Compute $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ for $\vec{F} = \langle x^2 \sin(z), y^2, xy \rangle$ and S the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane oriented upward.

Picture



Sol 2 (Direct Computation, $r\theta$-parameterization):

$\vec{S}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 1 - r^2 \rangle$ $(= \langle x, y, z \rangle)$

on $\{(r,\theta) : 0 \le r \le 1, \ 0 \le \theta \le 2\pi\}$

$\vec{S}_r = \langle \cos(\theta), \sin(\theta), -2r \rangle$

$\vec{S}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0 \rangle$

$$\vec{S}_r \times \vec{S}_\theta = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{bmatrix} = \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r \rangle$$

Should be same as in Sol 1.

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2\sin(z) & y^2 & xy \end{bmatrix} = \langle x - 0, -(y - x^2\cos(z)), 0 - 0 \rangle = \langle x, x^2\cos(z) - y, 0 \rangle$$

$$\therefore \text{curl}(\vec{F})(\vec{S}(r,\theta)) = \langle r\cos(\theta), r^2\cos^2(\theta)\cos(1 - r^2) - r\sin(\theta), 0 \rangle$$

$$\text{So} \quad \text{curl}(\vec{F})(\vec{S}(r,\theta)) \cdot (\vec{S}_r \times \vec{S}_\theta) = 2r^3\cos^2(\theta) + \left(2r^4\cos^2(\theta)\sin(\theta)\cos(1 - r^2) - 2r^3\sin^2(\theta)\right) + 0$$

$$= 2r^3\left(\left(\cos^2(\theta) - \sin^2(\theta)\right) + r\cos(1 - r^2)\cos^2(\theta)\sin(\theta)\right).$$

$$\therefore \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_D \text{curl}(\vec{F})(\vec{S}(r,\theta)) \cdot (\vec{S}_r \times \vec{S}_\theta) \, dA$$

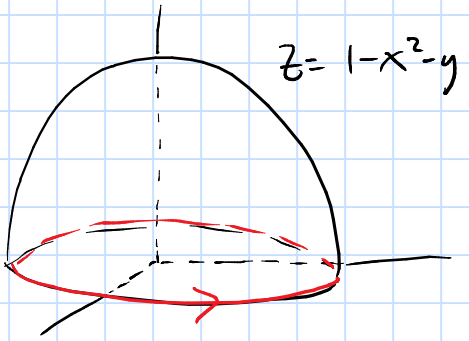
Proceed exactly as in Sol 1.

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} 2r^3\left(\left(\cos^2(\theta) - \sin^2(\theta)\right) + r\cos(1 - r^2)\cos^2(\theta)\sin(\theta)\right) d\theta \, dr$$

$$= 0 \qquad \blacksquare$$

Ex: Compute $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ for $\vec{F} = \langle x^2 \sin(z), y^2, xy \rangle$ and S the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane oriented upward.

Picture



Sol 3 (Stokes's Theorem):

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \int_{t=a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

∂S is the unit circle in the xy-plane

$\vec{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ on $0 \le t \le 2\pi$.

$\vec{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$

$\vec{F}(\vec{r}(t)) = \langle \cos^2(t)\sin(0), \sin^2(t), \cos(t)\sin(t) \rangle = \langle 0, \sin^2(t), \cos(t)\sin(t) \rangle$

$\therefore \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 0 + \sin^2(t)\cos(t) + 0 = \sin^2(t)\cos(t).$

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_{t=0}^{2\pi} \sin^2(t)\cos(t)\, dt$$

$\left(u = \sin(t), \quad u(2\pi) = 0; \quad du = \cos(t)\,dt, \quad u(0) = 0 \right)$

$$= \int_{u=0}^{0} u^2\, du = 0$$

(Nicest solution ☺)